REGULAR ALGEBRAS OF DIMENSION 4 WITH 3 GENERATORS

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ABSTRACT. We study Artin-Schelter regular algebras of global dimension 4 with three generators of degree one. We classify those which are domains and which have an additional $\mathbb{Z}^{\times 2}$-grading, and prove that all of these examples are also strongly noetherian, Auslander regular, and Cohen-Macaulay.

0. Introduction

In the late 1980's, Artin and Schelter introduced in [1] the notion of a *regular algebra* (now often called an Artin-Schelter regular or AS-regular algebra.) By definition, an AS-regular algebra is an $\mathbb{N}$-graded algebra $A = \bigoplus_{n \geq 0} A_n$ over a field k which is connected ($A_0 = k$) and satisfies the following three conditions: (i) A has finite global dimension d; (ii) A has polynomial growth; and (iii) A is Gorenstein in the sense that $\mathrm{Ext}_A^i(k, A) = 0$ for $i \neq d$ and $\mathrm{Ext}_A^d(k, A) \cong k$. (This last condition is now also called AS-Gorenstein.) Artin and Schelter's original work, together with subsequent work of Artin, Tate, and Van den Bergh, led to a complete classification of AS-regular algebras of global dimension 3, or equivalently, quantum $\mathbb{P}^2$s [1, 2, 3].

Since this seminal work there has been extensive research about AS-regular algebras of dimension four. The dimension of an AS-regular algebra means its global dimension. An important and famous example is the Sklyanin algebra of dimension 4, introduced by Sklyanin [15, 16] and studied further by Smith and Stafford [17], among others. Some other examples of well-known AS-regular algebras of dimension four include graded regular Clifford and skew-Clifford algebras [5]; homogenized $U(\mathfrak{sl}_2)$ [9]; deformations of the Sklyanin algebra [19]; skew polynomial rings in 4 variables; the quantum 2×2-matrix algebra [24]; AS-regular algebras with finitely many points; and AS-regular algebras containing a commutative quadric [20, 13, 14, 21, 22, 23]. Recently, Jun Zhang and the second author introduced the notion of a double Ore extension, which is also a useful method for constructing AS-regular algebras of dimension four [27, 28].

For simplicity, in this paper we only consider graded algebras that are generated in degree 1. By [11], every noetherian AS-regular algebra A of dimension four is generated by either 2, or 3, or 4 elements. If A is generated by 4 elements, then it is Koszul and its Hilbert series is equal to $(1-t)^{-4}$ (the same as the Hilbert series of the commutative polynomial ring $k[x_1, x_2, x_3, x_4]$). All algebras listed in the previous paragraph are generated by 4 elements. More recently, Lu, Palmieri, Wu, and the second author have investigated AS-regular algebras of dimension four that are generated by 2 elements [11].

The main objective of this paper is to begin to study AS-regular algebras A of dimension four that are generated by 3 elements. It is easy to find some examples of these by taking Ore extensions of 2-generated AS-regular algebras of dimension three, but few if any more general kinds of examples appear to be known. The idea of this paper is to look for some examples with the added special property that they have an additional $\mathbb{Z}^{\times 2}$-grading which is proper in the sense that $A_1 = A_{1,0} \oplus A_{0,1}$ with $A_{1,0} \neq 0$ and $A_{0,1} \neq 0$. We call those A which also have a normal regular element x of degree 1 such that $A/(x)$ is AS-regular of dimension three *normal extensions*. As normal extensions are in general easier to produce and to understand, we do not study them in detail here. By using some fairly Naïve techniques (primarily Bergman's diamond lemma [4]) and some simple code in the mathematical software program Maple to aid with the computation, we prove the following classification result.

Theorem 0.1. *Let A be an AS-regular domain of dimension 4 which is generated by 3 elements and properly $\mathbb{Z}^{\times 2}$-graded. Then either A is a normal extension, or else up to isomorphism A falls into one of eight 1 or 2-parameter families of examples (which we give explicitly in Section 3 below.)*

Since a noetherian AS-regular algebra of dimension 4 is a domain [11, Corollary 1.5], the above theorem also gives a classification of noetherian AS-regular algebras of dimension 4 which are generated by 3 elements and properly $\mathbb{Z}^{\times 2}$-graded. These eight families of algebras in Theorem 0.1 are generically pairwise non-isomorphic [Theorem 5.2(b)]. The known examples of AS-regular algebras of dimension 4 all have several other good ring-theoretic and homological properties which conjecturally may follow automatically from the AS-regularity assumption. One of our motivations is to continue to provide evidence for such a conjecture by showing that all of our examples have these properties. The following result follows from our explicit classification in Theorem 0.1.

Theorem 0.2. *Let A be a noetherian AS-regular algebra of dimension 4 that is generated by 3 elements. If A is properly $\mathbb{Z}^{\times 2}$-graded, then it is strongly noetherian, Auslander regular and Cohen-Macaulay.*

Combining this theorem with the results in [11, 28], we have the following corollary.

Corollary 0.3. *Let A be noetherian AS-regular algebra of dimension 4. If A is properly $\mathbb{Z}^{\times 2}$-graded, then it is strongly noetherian, Auslander regular and Cohen-Macaulay.*

Another motivation for our paper is to provide some new families of examples to serve as a testing ground for future questions and conjectures about AS-regular algebras. For example, Kirkman, Kuzmanovich, and the second author have developed a theory of rings of invariants of finite group actions on AS-regular algebras, concentrating mostly on the Koszul case [6, 7, 8], and having more examples at hand may be useful in extending this theory to the non-Koszul case. For this reason, we have included in our paper a careful determination of the automorphism groups of the generic examples in our classification [Theorem 5.2(a)].

1. **Preliminaries**

In this section, we give some background and lemmas that will be of use in our study of $\mathbb{Z}^{\times 2}$-graded AS-regular algebras with 3 generators. We work always over an algebraically closed field k of characteristic 0. All the algebras A in this paper will be connected graded, so $A = k \oplus A_1 \oplus A_2 \oplus \ldots$, and generated as a k-algebra by A_1. We write k for the trivial graded A-module $(A/A_{\geq 1})$ (as a left or right module, depending on context). The degree shift of a $\mathbb{Z}$-graded A module M is written as $M(d)$, where $M(d)_n = M_{n+d}$ for all n. If $\sigma : A \to A$ is a graded automorphism, recall that the *graded twist* of A by σ is the new algebra A^σ with the same underlying vector space as A and a new product defined by $x \star y = x\sigma^m(y)$, for $x \in A_m, y \in A_n$ (see [26] for details). By [26, Theorem 1.3], A and A^σ share many homological properties.

There are many standard techniques for relating the properties of a graded algebra A and a factor algebra $A/(x)$, where x is a homogeneous regular normal element, and these will play a large role in our analysis. We begin with a general such result that may be of independent interest.

Lemma 1.1. [18] *Let B be a connected graded algebra of global dimension $d < \infty$ and let $z \in B_1$ be a regular normal element of degree 1. Then $B/(z)$ has global dimension $d - 1$.*

Proof. Since B is connected graded, $\text{gldim } B = \max\{i \mid \underline{\text{Tor}}_i^B(k, k) \neq 0\}$, where $\underline{\text{Tor}}_i^B$ is the graded version of the Tor functor. Let $C = B/(z)$. We have the following graded version of the spectral sequence [12, Theorem 10.59]:

(E1.1.1) $$\underline{\text{Tor}}_p^C(k, \underline{\text{Tor}}_q^B(C, k)) \implies_p \underline{\text{Tor}}_n^B(k, k).$$

Since z is a regular element, by applying $\underline{} \otimes_B k$ to the short exact sequence $0 \to zB \to B \to C \to 0$ and considering the associated long exact sequence, we easily see that $\underline{\text{Tor}}_0^B(C, k) = k$, $\underline{\text{Tor}}_1^B(C, k) = k(-1)$, and $\underline{\text{Tor}}_i^B(C, k) = 0$ for $i > 1$. Hence the E^2-page of the spectral sequence (E1.1.1) has only two possibly non-zero rows; namely

$$q = 0 : \quad \underline{\text{Tor}}_p^C(k, k) \text{ for } p \geq 0, \quad \text{and} \quad q = 1 : \quad \underline{\text{Tor}}_p^C(k, k(-1)) \text{ for } p \geq 0.$$

Since (E1.1.1) converges, we have $\underline{\text{Tor}}_0^C(k, k) = \underline{\text{Tor}}_0^B(k, k)$ and a long exact sequence

$$\cdots \cdots \longrightarrow \underline{\text{Tor}}_4^C(k, k) \longrightarrow \underline{\text{Tor}}_2^C(k, k(-1))$$

(E1.1.2)
$$\longrightarrow \underline{\text{Tor}}_3^B(k, k) \longrightarrow \underline{\text{Tor}}_3^C(k, k) \longrightarrow \underline{\text{Tor}}_1^C(k, k(-1))$$

$$\longrightarrow \underline{\text{Tor}}_2^B(k, k) \longrightarrow \underline{\text{Tor}}_2^C(k, k) \longrightarrow \underline{\text{Tor}}_0^C(k, k(-1))$$

$$\longrightarrow \underline{\text{Tor}}_1^B(k, k) \longrightarrow \underline{\text{Tor}}_1^C(k, k) \longrightarrow 0.$$

By hypothesis, $d = \operatorname{gldim} B < \infty$. By (E1.1.2), $\underline{\operatorname{Tor}}^C_{i+1}(k,k) \cong \underline{\operatorname{Tor}}^C_{i-1}(k,k)(-1)$ for all $i > d$. Suppose for some $i > d$ that $\underline{\operatorname{Tor}}^C_{i+1}(k,k) \neq 0$. Then the minimal nonzero degree of $\underline{\operatorname{Tor}}^C_{i+1}(k,k)$ is equal to one more than the minimal nonzero degree of $\underline{\operatorname{Tor}}^C_{i-1}(k,k)$. On the other hand, note that the ith term of the minimal free resolution of the trivial C-module k_C is isomorphic to $\underline{\operatorname{Tor}}^C_i(k,k) \otimes_k C$. By the minimality of this free resolution, the minimal nonzero degree of $\underline{\operatorname{Tor}}^C_{i+1}(k,k)$ is at least one bigger than the minimal nonzero degree of $\underline{\operatorname{Tor}}^C_i(k,k)$, hence is at least two bigger than the minimal nonzero degree of $\underline{\operatorname{Tor}}^C_{i-1}(k,k)$. This is a contradiction, and thus for $i > d$, we must have $\underline{\operatorname{Tor}}^C_{i+1}(k,k) = 0$. Thus $\operatorname{gldim} C < \infty$. By [11, Lemma 7.6], $\operatorname{gldim} B = \operatorname{gldim} C + 1$. $\qquad\square$

The converse of the above lemma is also true, and in fact does not require the normal element to be in degree 1. The reader can find a proof in [11, Lemma 7.6], or may easily prove the converse using the same spectral sequence as in the lemma.

Corollary 1.2. *Let A be a connected graded algebra which is a domain. If there is a nonzero normal element $x \in A_1$, then $A/(x)$ is AS-regular if and only if A is.*

Proof. The preceding lemma and the converse in [11, Lemma 7.6] show that A has finite global dimension if and only if $A/(x)$ does. It is trivial that A has polynomial growth if and only if $A/(x)$ does, and that A is AS-Gorenstein if and only if $A/(x)$ is follows from the Rees Lemma, which says that $\operatorname{Ext}^i_A(k,A) \cong \operatorname{Ext}^{i-1}_{A/(x)}(k, A/(x))$. $\qquad\square$

Besides AS-regularity, some other important properties of graded algebras are the strong noetherian property, Auslander regularity, and the Cohen-Macaulay property. We will want to show that the examples we construct in this paper have these properties, but otherwise will not use them, so we refer the reader to [27, Section 5] for a review of the definitions. In the next result we review various techniques we will use to prove our examples have good properties.

Lemma 1.3. *Let A be a connected graded algebra. If any of the following conditions holds, then A is an AS-regular algebra which is also strongly noetherian, Auslander regular and Cohen-Macaulay.*

  (a) $A \cong B[t; \sigma, \delta]$ *for an AS-regular algebra B of dimension 3 with automorphism σ and σ-derivation δ.*
  (b) *A is a graded twist of B, where B is strongly noetherian, Auslander regular, and Cohen-Macaulay.*
  (c) *f is a regular homogeneous normal element and $B := A/(f)$ is AS-regular of dimension 3.*
  (d) *A has finite global dimension, and there are elements $f_1, \cdots, f_t$ such that the image of f_i is normal (not necessarily regular) in the factor ring $A/(f_1, \cdots, f_{i-1})$ for all i, and $A/(f_1, \cdots f_t)$ is finite dimensional over k.*

Proof. (a) This is [27, Lemma 5.3].
  (b) This follows from [27, Lemma 5.6].

(c,d) These follow from [27, Lemma 5.8], [27, Remark 5.2(b)] and Lemma 1.1. Part (d) is also a consequence of [25, Theorem 1]. □

2. Beginning analysis of $\mathbb{Z}^{\times 2}$-graded AS-regular algebras

Starting in this section, let A stand for an AS-regular algebra of dimension 4 with three generators of degree 1 which is properly $\mathbb{Z}^{\times 2}$-graded. Without loss of generality, we will write $A_1 = kx_1 + kx_2 + kx_3$ where $\deg x_1 = \deg x_2 = (1,0)$ and $\deg x_3 = (0,1)$. By [11, Proposition 1.4(b)], the Hilbert series of A as a connected $\mathbb{N}$-graded algebra is $h_A(t) = \dfrac{1}{(1-t)^3(1-t^2)}$. We always assume in this paper that A is a domain. (It is not known if this holds automatically for AS-regular algebras.) The assumptions on A in this paragraph are fixed for the rest of the paper.

Recall that an algebra A as above is a *normal extension* (of an AS-regular algebra of dimension three) if there is a nonzero normal element $z \in A_1$. In this case $A/(z)$ is 3-dimensional AS-regular algebra by Corollary 1.2, and so most homological and ring-theoretic properties of A follow automatically from those of $A/(z)$. Normal extensions are in general easier to understand; in the case of AS-regular algebras with 4-generators, for example, see [10] for an extensive study of them. Also, we do not need to know any details about those A which are normal extensions in order to prove Theorem 0.2. Thus for simplicity we will ignore normal extensions and classify only those A without normal elements in degree 1.

Since A is $\mathbb{Z}^{\times 2}$-graded, all the relations of A are also $\mathbb{Z}^{\times 2}$-homogeneous. In the next result we determine the multi-degrees of these relations.

Lemma 2.1. *Retain the hypotheses for A. Suppose that A has no normal element in degree 1.*

(a) *A has a relation in degree $(2,0)$ and a relation in degree $(1,1)$.*

(b) *A has a relation in degree $(1,2)$ and a relation in degree $(2,1)$.*

(c) *The $\mathbb{Z}^{\times 2}$-graded Hilbert series of A equals*

(E2.1.1) $$h_A(u,v) = \frac{1}{(1-u)^2(1-v)(1-uv)},$$

where u has degree $(1,0)$ and v has degree $(0,1)$.

Proof. By [11, Proposition 1.4(b)], A has two relations r_1, r_2 in degree 2 and two relations r_3, r_4 in degree 3. If there is a relation in degree $(0,2)$, then it must be $x_3^2 = 0$, which contradicts the fact A is a domain. If there are two relations in degree $(1,1)$, then using the fact A is a domain, after replacing them by linear combinations we can assume the two relations have the form

$$x_3x_1 = ax_1x_3 + bx_2x_3 \quad \text{and} \quad x_3x_2 = cx_1x_3 + dx_2x_3.$$

Thus x_3 will be a normal element, contradicting the hypothesis. By [6, Lemma 3.7(a)], A cannot have two relations in degree $(2,0)$. Thus A has one relation in degree $(2,0)$ and one relation in degree $(1,1)$, proving (a).

Consider now the two relations of degree 3. By [6, Lemma 3.7(a)], A cannot have relations in degree $(3,0)$, and since A is a domain, it has no relation of degree $(0,3)$. Thus the remaining relations have degree equal to $(2,1)$ or $(1,2)$. Let E be the Ext-algebra of A. Then

$$E = k \oplus E_{-1}^{\oplus 3} \oplus (E_{-2}^{\oplus 2} \oplus E_{-3}^{\oplus 2}) \oplus E_{-4}^{\oplus 3} \oplus E_{-5}$$

is in fact $\mathbb{Z} \times (\mathbb{Z}^{\times 2})$-graded. Using the language introduced in [11, Proposition 3.1(b)], $\{r_1^*, r_2^*\}$ is a basis of E_{-2}^2 and $\{r_3^*, r_4^*\}$ is a basis of E_{-3}^2. Since E is Frobenius [11, Theorem 1.9], the perfect pairing on the $\mathbb{Z} \times (\mathbb{Z}^{\times 2})$-graded algebra E implies that E_{-3}^2 has one element of degree $(2,1)$ and one element of degree $(1,2)$. This means that A has exactly one relation of each degree $(2,1)$ and $(1,2)$. Thus we have proved (b), and furthermore we have determined the multi-grading on E.

Finally, the multi-graded Hilbert series of A can be computed by using the multi-graded version of the free resolution in [11, Proposition 1.4(b)]. $\qquad\square$

We remark that if x_3 is a normal element in A (the case we threw away in the proof of the lemma), then the multi-graded Hilbert series of A is different from (E2.1.1).

Corollary 2.2. *If A has no normal nonzero element of degree 1, then the subalgebra C of A generated by x_1 and x_2 is a 2-dimensional AS-regular algebra.*

Proof. Since A is a domain, so is C. And C has at least one degree 2 relation, as we saw in the proof of the lemma. The assertion follows from [6, Lemma 3.7(a)]. $\qquad\square$

In the remainder of this section, we begin to analyze the possible relations $r_1 - r_4$ of non-normal-extensions A, and do some reductions to simply the form of the relations. By Lemma 2.1, without loss of generality from now on we can assume that

(E1.2.1) $\qquad\qquad \deg r_1 = (2,0), \ \deg r_2 = (1,1), \ \deg r_3 = (1,2), \text{ and } \deg r_4 = (2,1).$

Throughout our analysis, we will always let monomials be ordered with degree-lex order with $x_3 > x_2 > x_1$. Note that the opposite ring of A again satisfies the same hypotheses as A. One of our main tools in this paper will be Bergman's diamond lemma [4], with which we will assume the reader is familiar. We will sometimes pass to the opposite ring in order to simplify the relations. Since we will calculate the opposite rings of the examples we produce, we will not miss any examples this way.

The single relation of a 2-dimensional AS-regular algebra is well-known to be expressible in one of two forms. Since $C = k\langle x_1, x_2\rangle/(r_1)$ is AS-regular by the corollary above, by making a k-linear base change in $kx_1 + kx_2$, we can assume that r_1 is either a quantum-type relation $x_2 x_1 = p x_1 x_2$ for some $p \in k^\times$, or a Jordan-type relation $x_2 x_1 = x_1 x_2 + x_1^2$ (see the proof of [6, Lemma 3.7].) We write r_1 in a general form allowing both possibilities as follows:

$$r_1: \qquad x_2 x_1 = p x_1 x_2 + m x_1^2, \text{ where either } p \in k^\times \text{ and } m = 0, \text{ or } p = m = 1.$$

Next, r_2 has the general form $x_3(dx_2 + ax_1) + (nx_1 + bx_2)x_3 = 0$. First we claim that we may take $d = 1$. If $m = 0$, so that r_1 is of quantum type, then since A is a domain we have $(dx_2 + ax_1) \neq 0$. Then switching the roles of x_1 and x_2 if necessary, which does not affect the form of r_1, we may assume that $d \neq 0$. If r_1 is of Jordan type, so $m = p = 1$, then if $d = b = 0$ then x_1 will be normal in A. Since we are excluding normal extensions, by passing to the opposite ring if necessary, we can assume that $d \neq 0$. (Note that after passing to the opposite ring, we can put r_1 in the same form as before by replacing x_1 by $-x_1$.) Thus in all cases we may write the relation in the form

$$r_2 : \qquad x_3 x_2 = ax_3 x_1 + nx_1 x_3 + bx_2 x_3.$$

Further simplifications to this relation are possible here, depending on the case for r_1. We will make these simplifications in our case by case analysis later.

There is an overlap ambiguity $x_3 x_2 x_1$ between r_1 and r_2. Resolving this gives a new relation

$$r_5 : \qquad x_3 x_1 x_2 = qnx_1 x_3 x_1 + qbx_2 x_3 x_1 + zx_3 x_1^2,$$

where we write $q = p^{-1}$ and $z = qa - qm$ for notational convenience.

Since we have relations with leading terms $x_2 x_1$ and $x_3 x_2$, the degree $(1, 2)$ relation r_3 can be put into the form $(cx_1 + dx_2)x_3^2 + ex_3 x_1 x_3 + tx_3^2 x_1 = 0$. If $t = 0$, then as A is a domain, we must have $(cx_1 + dx_2)x_3 + ex_3 x_1 = 0$. Since r_2 is the only degree $(1, 1)$-relation by Lemma 2.1, this must be a scalar multiple of r_2, which it clearly is not. So $t \neq 0$, and the relation r_3 can be assumed to have the form

$$r_3 : \qquad x_3^2 x_1 = cx_1 x_3^2 + dx_2 x_3^2 + ex_3 x_1 x_3.$$

By Lemma 2.1, there is one more relation r_4 of degree $(2, 1)$. Given the leading terms of the relations we already have, r_4 can be written as

$$r_4 : \qquad (fx_1^2 + gx_1 x_2 + hx_2^2)x_3 + (jx_1 + kx_2)x_3 x_1 + \ell x_3 x_1^2 = 0.$$

The leading term of r_4 is not clear at this point. We will do a separate analysis of each possible case of the leading term of r_4, but the only case in which we find any AS-regular algebras will be when $\ell \neq 0$.

3. Classification of $\mathbb{Z}^{\times 2}$-graded AS-regular algebras with 3 generators

We continue to assume the notation and hypotheses of the preceding section, so that A is a $\mathbb{Z}^{\times 2}$-graded AS-regular algebra with 3 generators which is a domain and not a normal extension.

3.1. **Case $\ell \neq 0$.** We assume now that $\ell \neq 0$, so the leading term of r_4 is $x_3 x_1^2$. Then we can take $\ell = -1$ and by the reductions in the previous section, we assume that A is presented by the following four minimal

relations:

$$r_1 : x_2x_1 = px_1x_2 + mx_1^2 \qquad (p \neq 0 \text{ and } m = 0, \text{ or } p = 1 = m)$$

$$r_2 : x_3x_2 = ax_3x_1 + nx_1x_3 + bx_2x_3$$

$$r_3 : x_3^2x_1 = cx_1x_3^2 + dx_2x_3^2 + ex_3x_1x_3$$

$$r_4 : x_3x_1^2 = (fx_1^2 + gx_1x_2 + hx_2^2)x_3 + (jx_1 + kx_2)x_3x_1.$$

Recall that the overlap between r_1 and r_2 also gives the relation

$$r_5 : x_3x_1x_2 = qnx_1x_3x_1 + qbx_2x_3x_1 + zx_3x_1^2,$$

where $q = p^{-1}$ and $z = qa - qm$.

Lemma 3.1. *The set of monomials $\{x_1^i x_2^j (x_3x_1)^k x_3^l \mid i, j, k, l \geq 0\}$ is a k-basis for A, and all overlap ambiguities among $r_1 - r_5$ resolve.*

Proof. Recall that since A is AS-regular, the Hilbert series of A is $h_A(t) = \dfrac{1}{(1-t)^3(1-t^2)}$. The set of monomials not containing a leading term of any relation $r_1 - r_5$ is $\{x_1^i x_2^j (x_3x_1)^k x_3^l \mid i, j, k, l \geq 0\}$, and so this spans A as a k-vector space. Since this set of monomials already has the correct Hilbert series, it is immediate from Bergman's diamond lemma that these monomials are a k-basis and so all overlaps resolve. □

Given the lemma above, as a necessary condition for AS-regularity we seek conditions on the coefficients of the r_i for which all overlaps resolve. We use a simple program in Maple to help us calculate the coefficients of the new relations that these overlaps produce.

There are three overlaps to be resolved, $x_3^2x_1x_2$ between r_3 and r_5, $x_3^2x_1^2$ between r_3 and r_4, and $x_3x_1x_2x_1$ between r_5 and r_1. The respective three new relations are the following:

$$
\begin{aligned}
r_6 : \quad & (-qba - qn + ea - ze)x_3x_1x_3x_1 \\
& + (enj + cae + cba + cn - zce + ebzj)x_1x_3x_1x_3 \\
& + (enk + dba + dae + dn - zde + ebzk)x_2x_3x_1x_3 \\
& + (enf + cbn + ac^2 - zc^2 + dacm + dbnm - qbnc - qb^2cm + ebzf - zdcm)x_1^2x_3^2 \\
& + (eng + cad + dacp + dbnp - qbnd - zcd + ebzg - zdcp)x_1x_2x_3^2 \\
& + (enh - qb^2d - zd^2 + ad^2 + db^2 + ebzh)x_2^2x_3^2 = 0,
\end{aligned}
$$

$$r_7: \qquad (-ka + e - j)x_3x_1x_3x_1$$

$$+(-fj + ce - gzj - gqn - hna - enk - haqn - hazj)x_1x_3x_1x_3$$

$$+(de - ebk - gzk - gqb - fk - hba - haqb - hazk)x_2x_3x_1x_3$$

$$+(-hn^2 + dcm + c^2 - gzf - f^2 - knc - hazf - hbnm - kbcm)x_1^2x_3^2$$

$$+(dcp + cd - fg - zg^2 - hnb - knd - hazg - hbnp - kbcp)x_1x_2x_3^2$$

$$+(-hb^2 + d^2 - gzh - fh - azh^2 - kbd)x_2^2x_3^2 = 0,$$

$$r_8: \qquad (-zf + mj^2 + mf + pzj^2 + pfa$$

$$-jqn - zj^2 - kzjm + jn + kjm^2 + knm + pkzjm - qbjm)x_1^2x_3x_1$$

$$+(-zg + mg + pga - jzk + mjk - qnk + knp + kzjp^2 + mkjp)x_1x_2x_3x_1$$

$$+(mh + mk^2 - zh + pha - kqb - zk^2 + kb + pzk^2)x_2^2x_3x_1$$

$$+(pfn - jzf + hnpm^2 - kzfm + pgnm + pjzf + mjf$$

$$+kfm^2 - qnf + hnp^2m^2 + kzfp^2m + kfpm^2 - qbfpm - qbfm)x_1^3x_3$$

$$+(gnp^2 + pfb - gzj + hnmp^2 - kzfp^2 - kzgm + pjzg + mjg$$

$$+kgm^2 - gqn + hnmp^3 + kzfp^3 + pkzgm + mkfp^2 - bfp - qbgm)x_1^2x_2x_3$$

$$+(hnp^3 + pgb - jzh - kzgp + pjzh + mjh - qnh + kzgp^2 + mkgp - qbgp)x_1x_2^2x_3$$

$$+(phb - kzh + pkzh + mkh - qbh)x_2^3x_3 = 0.$$

We conclude that if A is AS-regular then all of the coefficients of r_6, r_7, and r_8 are 0. We will use Maple to help solve for the possible solutions to the system of equations given by setting the coefficients of r_6, r_7, r_8 to 0, together with $pq = 1$, $z = qa - qm$. It is helpful to break this up into three cases, where in each case we can eliminate some parameters to make the solution simpler.

3.2. **Case $\ell \neq 0$, $m = 0, p = 1$.** This is the special case of the quantum-type r_1 relation where x_1 and x_2 commute. In this case, we are free to make any change of basis in $kx_1 + kx_2$ without affecting r_1. Replacing x_2 by $x_2 - ax_1$, we can assume that r_2 has the form $x_3x_2 = nx_1x_3 + bx_2x_3$. If $n = 0$, then x_2 is a normal element and we exclude this case. Thus we may assume that $n \neq 0$; replacing x_1 by $nx_1 + bx_2$, our relation becomes $x_3x_2 = x_1x_3$. In conclusion, we may assume that $a = b = 0$ and $n = 1$.

Thus we seek to solve the system given by setting the coefficients of $r_6 - r_8$ to 0, under the conditions $m = 0, p = q = 1, z = a = b = 0, n = 1$. It is easy to see without a computer that there are no solutions: the coefficient of $x_3x_1x_3x_1$ in r_6 is $-qba - ze - qn + ea$, which in this case is equal to -1.

3.3. **Case $\ell \neq 0$, $m = 0, p \neq 1$.** This is the generic case of the quantum-type r_1 relation, and is the case in which we will find almost all of the examples of AS-regular algebras. We make some reductions to the form

of r_2. If $n = 0$, then $a \neq 0$ (otherwise else x_2 is normal) and $b \neq 0$ (or else A is not a domain). Thus passing to the opposite ring if necessary we may assume $n \neq 0$. Further, replacing x_1 by nx_1, we may assume that the relation has the form $r_2 : x_3x_2 = ax_3x_1 + x_1x_3 + bx_2x_3$.

Before solving the system of equations given by setting the coefficients of $r_6 - r_8$ equal to 0, we give a series of lemmas that will be helpful to simplify our analysis. First, some of the solutions will lead to non-domains. The following lemma will help us avoid these outright.

Lemma 3.2. *Suppose that $n = 1$. If A is a domain, then $d \neq bc$.*

Proof. The relation r_2 gives $x_1x_3 = x_3x_2 - ax_3x_1 - bx_2x_3$. Substituting this into the $x_1x_3^2$ term of r_3 gives $x_3^2x_1 = cx_3x_2x_3 + (d-bc)x_2x_3^2 + (e-ac)x_3x_1x_3$. If $d = bc$, then we see that $x_3(x_3x_1 - cx_2x_3 + (ac-e)x_1x_3) = 0$ is a relation. As A is a domain, this forces the relation $x_3x_1 - cx_2x_3 + (ac - e)x_1x_3 = 0$. However, r_2 is the only relation of degree $(1, 1)$ by Lemma 2.1, and this is a contradiction. $\qquad\square$

Next, we note that we can twist away the coefficient b.

Lemma 3.3. *Let $A = A(b) = A\langle x_1, x_2, x_3 \rangle / (r_1, r_2, r_3, r_4)$, where $b \neq 0$ and*

$$r_1 : x_2x_1 = px_1x_2$$

$$r_2 : x_3x_2 = (a/b)x_3x_1 + x_1x_3 + bx_2x_3$$

$$r_3 : x_3^2x_1 = b^2cx_1x_3^2 + b^3dx_2x_3^2 + bex_3x_1x_3$$

$$r_4 : x_3x_1^2 = b^2fx_1^2x_3 + b^3gx_1x_2x_3 + b^4hx_2^2x_3 + bjx_1x_3x_1 + b^2kx_2x_3x_1.$$

Then $A(b)$ is isomorphic to a graded twist of $A(1)$.

Proof. The ring $A(1)$ has a automorphism ϕ defined by $x_1 \to x_1, x_2 \to x_2, x_3 \to b^{-1}x_3$. If B is the graded twist of $A(1)$ by this automorphism, then the relations of B can be found by multiplying each term in the relations of $A(1)$ by an appropriate power of b. A change of variable replacing x_1 by bx_1 can then be checked to produce the relations of $A(b)$. $\qquad\square$

For some of our examples, we will have to resort to giving an explicit free resolution of k to prove that the example has finite global dimension. The following lemma will cover those cases.

Lemma 3.4. *Suppose that $\{x_1^i x_2^j (x_3x_1)^k x_3^l \mid i, j, k, l \geq 0\}$ is a k-linear basis of A, and that there is a complex of left modules the form*

(E3.4.1) $\qquad 0 \to A(-5) \xrightarrow{d_4} A(-4)^{\oplus 3} \xrightarrow{d_3} A(-3)^{\oplus 2} \oplus A(-2)^{\oplus 2} \xrightarrow{d_2} A(-1)^{\oplus 3} \xrightarrow{d_1} A \xrightarrow{d_0} {}_Ak \to 0,$

where the elements of the free modules are row vectors, and each d_i is given by right multiplication by a matrix. We assume that

(i) $d_4 = (x_1, x_2, x_3)$;

(ii) $d_1 = (x_1, x_2, x_3)^t$ and d_2 is constructed from the relations $r_1 - r_4$; and

(iii) d_3 is given by a matrix of the form $M := \begin{pmatrix} \alpha x_3^2 & * & * & \beta x_3 \\ \phi x_3^2 & * & * & \varphi x_3 \\ -x_3 x_1 + y x_3 & * & * & * \end{pmatrix}$, where $\alpha\varphi - \beta\phi \neq 0$ and

$y \in kx_1 + kx_2 + kx_3$.

Then (E3.4.1) is exact and A has global dimension 4.

Proof. It is clear from the form of the k-basis for A that x_3 is a right nonzerodivisor; thus d_4 is injective. We claim that to prove (E3.4.1) is exact, we need only show that $\ker d_3 = \operatorname{im} d_4$. If this equation holds, then there is only one possible nonzero cohomology of the complex X given by (E3.4.1), namely $H^2(X)$. Since X is a complex, the Hilbert series of the terms satisfy $\sum_i (-1)^i h_{H^i(X)}(t) = \sum_i (-1)^i h_{X^i}(t) = 0$, where the latter equality follows from the assumed Hilbert series of A. This implies that $H^2(X) = 0$ and hence X is exact, proving the claim.

Now suppose that $(v_1, v_2, v_3) \in \ker d_3$. Then $\alpha v_1 x_3^2 + \phi v_2 x_3^2 + v_3(-x_3 x_1 + y x_3) = 0$. Express v_3 as a linear combination of $x_1^i x_2^j (x_3 x_1)^k x_3^l$. Then $\alpha v_1 x_3^2 + \phi v_2 x_3^2 + v_3(-x_3 x_1 + y x_3) = 0$ implies that $v_3 = w x_3$ for some w. Working modulo $\operatorname{im} d_4$ and replacing (v_1, v_2, v_3) by $(v_1 - w x_1, v_2 - w x_2, v_3 - w x_3)$, we may assume that $v_3 = 0$. Therefore we have $\alpha v_1 x_3^2 + \phi v_2 x_3^2 = 0$. Since x_3 is a right nonzerodivisor, $\alpha v_1 + \phi v_2 = 0$. Similarly, by the fourth column of the equation $(v_1, v_2, v_3)M = 0$, we have $\beta v_1 + \varphi v_2 = 0$. Since $\alpha\varphi - \beta\phi \neq 0$, we have $v_1 = v_2 = 0$. This shows that $(v_1, v_2, v_3) \in \operatorname{im} d_4$. It follows that $\ker d_3 = \operatorname{im} d_4$ and so X is exact by the argument above. Since X is the free resolution of the trivial module k, A has global dimension four. $\square$

We are now ready to solve the system given by setting the coefficients of $r_6 - r_8$ equal to 0, under the conditions $m = 0, p \neq 1, n = 1$. By Lemma 3.2, since we want A to be a domain, we can also assume that $d \neq bc$. By Lemma 3.3, if $b \neq 0$ then some twist-equivalent algebra has $b = 1$. Thus it suffices to assume that $b = 0$ or $b = 1$. In the latter case (which holds for all solutions we find) we will twist the b back and include it as a parameter when we present the solutions. Solving the system with the constraints listed above gives 8 families of solutions, all of which turn out to be AS-regular, and which we list in the following pages along with some of their important properties. We state many of the properties of these algebras without further proof. Any claims about twist equivalence follow from Lemma 3.3, and the other properties are generally checked by a tedious but straightforward repeated application of the relations. See Remark 3.14 for more about our computational methods.

Example 3.5. Let $\mathcal{A}(b, q)$ be the algebra with the relations

$$r_1 : x_2 x_1 = \frac{1}{q} x_1 x_2 \quad (q \neq 1)$$

$$r_2 : x_3 x_2 = -\frac{1}{q^2 b} x_3 x_1 + x_1 x_3 + b x_2 x_3 \quad (b \neq 0)$$

$$r_3 : x_3^2 x_1 = -q^3 b^2 x_1 x_3^2 + (q^2 + q) b x_3 x_1 x_3$$

$$r_4 : x_3 x_1^2 = -q^3 b^2 x_1^2 x_3 + (q^2 + q) b x_1 x_3 x_1.$$

(3.5.1) $\mathcal{A}(b, q)$ is isomorphic to a graded twist of $\mathcal{A}(1, q)$ [Lemma 3.3].

(3.5.2) The opposite ring of $\mathcal{A}(b, q)$ is isomorphic to $\mathcal{A}(1/b, 1/q)$.

(3.5.3) There is a graded algebra isomorphism $\sigma : \mathcal{A}(b, q) \to \mathcal{A}(q^2 b, q^{-1})$ determined by

$$\sigma : x_1 \to b x_2, x_2 \to q^{-2} b^{-1} x_1, x_3 \to x_3.$$

(3.5.4) The element $y := x_3 x_1 - q^2 b x_1 x_3$ is normal in $\mathcal{A}(b, q)$ and $\mathcal{A}(b, q)/(y)$ is an AS-regular algebra of dimension 3.

Let $\mathcal{A} = \mathcal{A}(b, q)$ for some fixed b, q and let y be the normal element given above. Because we know that $h_{\mathcal{A}}(t) = \dfrac{1}{(1 - t)^3(1 - t^2)}$ by construction and $h_{\mathcal{A}/(y)}(t) = \dfrac{1}{(1 - t^3)}$, the normal element y must be regular. Thus $\mathcal{A}$ satisfies the condition in Lemma 1.3(c); in particular, it is AS-regular and has the other good properties listed there.

The algebra $\mathcal{A}$ is also an iterated Ore extension. Let B be the subalgebra generated by x_1, x_3 and $y = x_3 x_1 - q^2 b x_1 x_3$. Then

$$B = k\langle x_1, y, x_3\rangle/(y x_1 - q b x_1 y, x_3 y - q b x_3 y, x_3 x_1 - q^2 b x_1 x_3 - y)$$

is an iterated Ore extension, hence it is AS-regular of dimension 3. Now one may check that $\mathcal{A} \cong B[x_2; \phi, \delta]$, where $\sigma : x_1 \to \dfrac{1}{q} x_1, x_3 \to \dfrac{1}{b} x_3$ and $\delta : x_1 \to 0, x_3 \to \dfrac{1}{q^2 b^2} x_3 x_1 - \dfrac{1}{b} x_1 x_3$. Thus Lemma 1.3(a) also applies to the algebra $\mathcal{A}$.

Example 3.6. Let $\mathcal{B}(b)$ denote the algebra with relations

$$r_1 : x_2 x_1 = -x_1 x_2$$

$$r_2 : x_3 x_2 = -\frac{1}{b} x_3 x_1 + x_1 x_3 + b x_2 x_3 \quad (b \neq 0)$$

$$r_3 : x_3^2 x_1 = b^2 x_1 x_3^2$$

$$r_4 : x_3 x_1^2 = -b^3 x_1 x_2 x_3 + b x_1 x_3 x_1 + b^2 x_2 x_3 x_1.$$

(3.6.1) $\mathcal{B}(b)$ is isomorphic to a graded twist of $\mathcal{B}(1)$.

(3.6.2) The opposite ring of $\mathcal{B}(b)$ is isomorphic to $\mathcal{B}(1/b)$.

(3.6.3) There is a graded algebra automorphism of $\mathcal{B}(b)$ determined by

$$\sigma : x_1 \to b x_2, x_2 \to b^{-1} x_1, x_3 \to x_3.$$

(3.6.4) The element $y := x_3x_1 - b^2x_2x_3$ is a normal element, and the factor algebra $\mathcal{B}(b)/(y)$ is a 3-dimensional AS-regular algebra generated by 3 elements. The element y is regular by the same proof as for the family $\mathcal{A}$, and so $\mathcal{B}(b)/(y)$ satisfies Lemma 1.3(c). Therefore $\mathcal{B}(b)$ has the properties listed in Lemma 1.3.

Example 3.7. Let $\mathcal{C}(b)$ denote the algebra with relations

$$r_1 : x_2x_1 = -x_1x_2$$

$$r_2 : x_3x_2 = -\frac{1}{b}x_3x_1 + x_1x_3 + bx_2x_3 \quad (b \neq 0)$$

$$r_3 : x_3^2x_1 = b^2x_1x_3^2$$

$$r_4 : x_3x_1^2 = b^3x_1x_2x_3 + bx_1x_3x_1 + b^2x_2x_3x_1.$$

(3.7.1) $\mathcal{C}(b)$ is isomorphic to a graded twist of $\mathcal{C}(1)$.

(3.7.2) The opposite ring of $\mathcal{C}(b)$ is isomorphic to $\mathcal{C}(1/b)$.

(3.7.3) There is a graded algebra automorphism of $\mathcal{C}(b)$ determined by

$$\sigma : x_1 \to bx_2, x_2 \to b^{-1}x_1, x_3 \to x_3.$$

(3.7.4) The element $y := x_3x_1 - bx_1x_3$ is a normal element, and the factor algebra $\mathcal{C}(b)/(y)$ is a 3-dimensional AS-regular algebra generated by 3 elements. Similarly as above, y is a regular element and Lemma 1.3(c) holds.

Example 3.8. Let $\mathcal{D}(b,h)$ be the algebra with relations

$$r_1 : x_2x_1 = -x_1x_2$$

$$r_2 : x_3x_2 = -\frac{1}{b}x_3x_1 + x_1x_3 + bx_2x_3 \quad (b \neq 0)$$

$$r_3 : x_3^2x_1 = b^2x_1x_3^2$$

$$r_4 : x_3x_1^2 = \left(\frac{h}{b^2} - b^2\right)x_1^2x_3 + hx_2^2x_3.$$

(3.8.1) $\mathcal{D}(b,h)$ is isomorphic to a graded twist of $\mathcal{D}(1, hb^{-4})$.

(3.8.2) The opposite ring of $\mathcal{D}(b,h)$ is isomorphic to $\mathcal{D}(1/b, hb^{-8})$.

(3.8.3) There is a graded algebra isomorphism $\sigma : \mathcal{D}(b,h) \to \mathcal{D}(b, 2b^4 - h)$ determined by

$$\sigma : x_1 \to bx_2, x_2 \to b^{-1}x_1, x_3 \to x_3.$$

We need further analysis to see why the algebra is AS-regular. Following Lemma 1.3(b), we may assume that $b = 1$. We construct a potential free resolution of $\mathcal{D} := \mathcal{D}(1, h)$ of the form

$$0 \to \mathcal{D}(-5) \xrightarrow{d_4} \mathcal{D}(-4)^{\oplus 3} \xrightarrow{d_3} \mathcal{D}(-3)^{\oplus 2} \oplus \mathcal{D}(-2)^{\oplus 2} \xrightarrow{d_2} \mathcal{D}(-1)^{\oplus 3} \xrightarrow{d_1} \mathcal{D} \xrightarrow{d_0} {}_\mathcal{D}k \to 0$$

where here

$$d_3 : \begin{pmatrix} x_3^2 & -x_1 x_3 & -x_2 & x_3 \\ 0 & 0 & -x_1 & x_3 \\ -x_3 x_1 & (h-1)x_1^2 + h x_2^2 & 0 & -x_1 - x_2 \end{pmatrix},$$

$$d_2 : \begin{pmatrix} -x_2 & -x_1 & 0 \\ -x_3 & -x_3 & x_1 + x_2 \\ -x_3^2 & 0 & x_1 x_3 \\ -x_3 x_1 & 0 & (h-1)x_1^2 + h x_2^2 \end{pmatrix},$$

and the other maps are as in Lemma 3.4. It is straightforward to check that this is a complex, and it is then a free resolution of k by Lemma 3.4. Thus $\mathcal{D}$ has global dimension 4.

We also need to show that $\mathcal{D}$ has enough normal elements. It follows from the relations r_1 and r_2 that $x_1^2 + x_2^2$ is central in $\mathcal{D}$, and it follows from the relations r_2 and r_3 that x_3^2 is central. Using the relations r_1 and r_4, one sees that x_1^2 is central in $B := \mathcal{D}/(x_1^2 + x_2^2, x_3^2)$. Let C be the factor ring $B/(x_1^2)$. Then $x_3 x_1 - x_1 x_3$ is normal in C. Finally, given the leading terms we have, one may check that $C/(x_3 x_1 - x_1 x_3)$ is spanned by $\{1, x_1, x_2, x_3, x_1 x_2, x_1 x_3, x_2 x_3, x_1 x_2 x_3\}$, and so is finite-dimensional. Lemma 1.3(d) now applies to $\mathcal{D}$, and using also Lemma 1.3(b), $\mathcal{D}(b, h)$ is strongly noetherian, Auslander regular and Cohen-Macaulay.

Example 3.9. Let $\mathcal{E}(b, \gamma)$ be the algebra with relations

$$r_1 : x_2 x_1 = -x_1 x_2$$

$$r_2 : x_3 x_2 = -\frac{1}{b} x_3 x_1 + x_1 x_3 + b x_2 x_3 \quad (b \neq 0)$$

$$r_3 : x_3^2 x_1 = b^3 x_2 x_3^2$$

$$r_4 : x_3 x_1^2 = \gamma b^3 x_1 x_2 x_3 + b x_1 x_3 x_1 + b^2 x_2 x_3 x_1 \quad (\gamma = \pm\sqrt{-1}.)$$

(3.9.1) $\mathcal{E}(b, \gamma)$ is isomorphic to a graded twist of $\mathcal{E}(1, \gamma)$.

(3.9.2) The opposite ring of $\mathcal{E}(b, \gamma)$ is isomorphic to $\mathcal{E}(b^{-1}, \gamma^{-1})$.

(3.9.3) There is a graded algebra automorphism of $\mathcal{E}(b, \gamma)$ determined by

$$\sigma : x_1 \to b x_2, x_2 \to b^{-1} x_1, x_3 \to x_3.$$

In this and the remaining examples in this case, the proof of AS-regularity is by the same method as in Example 3.8: we twist away b and then construct an explicit free resolution of k and a sequence of normal elements. Let $\mathcal{E} := \mathcal{E}(1, \gamma)$. One may check using Lemma 3.4 that $_\mathcal{E} k$ has a free resolution of the form given

in that lemma, where

$$d_3 : \begin{pmatrix} -x_3^2 & x_2x_3 & -x_2 & -x_3 \\ 0 & 0 & -x_1 & -x_3 \\ x_1x_3 + x_2x_3 - x_3x_1 & \gamma x_1x_2 & 0 & x_1 + x_2 \end{pmatrix},$$

$$d_2 : \begin{pmatrix} -x_2 & -x_1 & 0 \\ -x_3 & -x_3 & x_1 + x_2 \\ -x_3^2 & 0 & x_2x_3 \\ -x_3x_1 + x_1x_3 + x_2x_3 & 0 & \gamma x_1x_2 \end{pmatrix}.$$

Further, $\mathcal{E}$ has normal elements $x_1x_2, x_3^2, x_1^2 + x_2^2$. It is easily checked that $B := \mathcal{E}/(x_1x_2, x_3^2, x_1^2 + x_2^2)$ has a normal element $x_3x_1 - x_2x_3$ and that $B/(x_3x_1 - x_2x_3)$ is finite dimensional. Now Lemma 1.3(b,d) applies to show $\mathcal{E}(b, \gamma)$ is strongly noetherian, Auslander regular and Cohen-Macaulay.

Example 3.10. Let $\mathcal{F}(b, \gamma)$ denote the algebra with relations

$$r_1 : x_2x_1 = \gamma^2 x_1x_2 \quad (\gamma = \text{primitive } \sqrt[3]{1})$$

$$r_2 : x_3x_2 = -\frac{\gamma}{b}x_3x_1 + x_1x_3 + bx_2x_3 \quad (b \neq 0)$$

$$r_3 : x_3^2x_1 = \gamma^2 b^3 x_2x_3^2 - bx_3x_1x_3$$

$$r_4 : x_3x_1^2 = \gamma b^3 x_1x_2x_3 + \gamma^2 b^2 x_2x_3x_1.$$

(3.10.1) $\mathcal{F}(b, \gamma)$ is isomorphic to a graded twist of $\mathcal{F}(1, \gamma)$.

(3.10.2) The opposite ring of $\mathcal{F}(b, \gamma)$ is isomorphic to $\mathcal{F}(b^{-1}, \gamma^{-1})$.

As in the previous examples, we let $\mathcal{F} = \mathcal{F}(1, \gamma)$, and there is a free resolution of $_{\mathcal{F}}k$ satisfying Lemma 3.4 with

$$d_3 : \begin{pmatrix} \gamma^2 x_3^2 & -\gamma x_3x_1 + x_2x_3 & -\gamma x_2 & x_3 \\ 0 & 0 & \gamma^2 x_1 & x_3 \\ -\gamma^2 x_2x_3 + x_3x_1 & x_1x_2 & 0 & -\gamma x_1 - x_2 \end{pmatrix},$$

$$d_2 : \begin{pmatrix} -x_2 & \gamma^2 x_1 & 0 \\ -\gamma x_3 & -x_3 & x_1 + x_2 \\ -x_3^2 & 0 & \gamma^2 x_2x_3 - x_3x_1 \\ \gamma^2 x_2x_3 - x_3x_1 & 0 & \gamma x_1x_2 \end{pmatrix}.$$

The algebra $\mathcal{F}$ has no normal elements of degree 1 and 2, but has three normal elements of degree 3: $x_1^3 + x_2^3, x_3^3$ and $x_1^2 x_2$. Let $B := \mathcal{F}/(x_1^3 + x_2^3, x_3^3, x_1^2 x_2)$. Then B has a normal element

$$d = (x_3x_1)^3 + x_1(x_3x_1)^2 x_3 - \gamma^2 x_2(x_3x_1)^2 x_3 - x_1^2(x_3x_1)x_3^2 - x_1x_2(x_3x_1)x_3^2$$

and $B/(d)$ is finite dimensional over k. Now Lemma 1.3(b,d) applies to show $\mathcal{F}(b, \gamma)$ is AS-regular with the usual good properties.

Example 3.11. Let $\underline{\mathcal{F}}(b, \gamma)$ be the algebra with relations

$$r_1 : x_2 x_1 = \gamma^2 x_1 x_2 \quad (\gamma = \text{primitive } \sqrt[3]{1})$$

$$r_2 : x_3 x_2 = -\frac{\gamma}{b} x_3 x_1 + x_1 x_3 + b x_2 x_3 \quad (b \neq 0)$$

$$r_3 : x_3^2 x_1 = \gamma^2 b^2 x_1 x_3^2 - b^3 x_2 x_3^2 - b x_3 x_1 x_3$$

$$r_4 : x_3 x_1^2 = -b^4 x_2^2 x_3 + \gamma^2 b x_1 x_3 x_1 - b^2 x_2 x_3 x_1.$$

(3.11.1) $\underline{\mathcal{F}}(b, \gamma)$ is isomorphic to a graded twist of $\underline{\mathcal{F}}(1, \gamma)$.

(3.11.2) The opposite ring of $\underline{\mathcal{F}}(b, \gamma)$ is isomorphic to $\underline{\mathcal{F}}(b^{-1}, \gamma^{-1})$.

(3.11.3) There is a graded algebra isomorphism $\sigma : \mathcal{F}(b, \gamma) \to \underline{\mathcal{F}}(\gamma^2 b, \gamma^2)$ determined by

$$\sigma : x_1 \to b x_2, x_2 \to \gamma b^{-1} x_1, x_3 \to x_3.$$

The isomorphism above shows why we have named this family $\underline{\mathcal{F}}(b, \gamma)$. Since the previous example proved that $\mathcal{F}(b, \gamma)$ has all desired properties, we need not study this example further.

Example 3.12. Let $\mathcal{G}(b, \gamma)$ denote the algebra with relations

$$r_1 : x_2 x_1 = -x_1 x_2$$

$$r_2 : x_3 x_2 = -\frac{1}{b} x_3 x_1 + x_1 x_3 + b x_2 x_3 \quad (b \neq 0)$$

$$r_3 : x_3^2 x_1 = b^3 x_2 x_3^2$$

$$r_4 : x_3 x_1^2 = \frac{b^2}{2\gamma} x_1^2 x_3 + \gamma b^4 x_2^2 x_3 \quad (\gamma = \frac{1 \pm i}{2}).$$

(3.12.1) $\mathcal{G}(b, \gamma)$ is isomorphic to a graded twist of $\mathcal{G}(1, \gamma)$.

(3.12.2) The opposite ring of $\mathcal{G}(b, \gamma)$ is isomorphic to $\mathcal{G}(b^{-1}, \overline{\gamma})$.

(3.12.3) There is a graded algebra automorphism of $\mathcal{G}(b, \gamma)$ determined by

$$\sigma : x_1 \to b x_2, x_2 \to b^{-1} x_1, x_3 \to x_3.$$

Similarly as in the previous examples, Setting $\mathcal{G} = \mathcal{G}(1, \gamma)$, there is a free resolution of $_{\mathcal{G}}k$ satisfying Lemma 3.4, where

$$d_3 : \begin{pmatrix} 0 & 0 & -x_2 & x_3 \\ x_3^2 & -x_2 x_3 & -x_1 & x_3 \\ -x_3 x_1 & \bar{\gamma} x_1^2 + \gamma x_2^2 & 0 & -x_1 - x_2 \end{pmatrix}, \quad d_2 : \begin{pmatrix} -x_2 & -x_1 & 0 \\ -x_3 & -x_3 & x_1 + x_2 \\ -x_3^2 & 0 & x_2 x_3 \\ -x_3 x_1 & 0 & \bar{\gamma} x_1^2 + \gamma x_2^2 \end{pmatrix}.$$

One can check that $\mathcal{G}$ has normal elements $x_3^2, x_1^2 + x_2^2$. Set $B := \mathcal{G}/(x_3^2, x_1^2 + x_2^2)$. Then B has a normal element x_1^2, and $C := B/(x_1^2)$ has normal elements $x_3 x_1 - x_1 x_3$ and $x_3 x_2 - x_2 x_3$. Finally, the factor ring $C/(x_3 x_1 - x_1 x_3, x_3 x_2 - x_2 x_3)$ is finite dimensional. This is sufficient to show that $\mathcal{G}(b, \gamma)$ is strongly noetherian, Auslander regular and Cohen-Macaulay by Lemma 1.3(b,d).

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3.4. **Case $\ell \neq 0$, $p = m = 1$.** This is the case of the Jordan type r_1, $x_2 x_1 = x_1 x_2 + x_1^2$. We make a reduction to r_2. Replacing x_2 by $a x_1 + x_2$ (which does not change the relation r_1), we can assume that r_2 has the form $x_3 x_2 = n x_1 x_3 + b x_2 x_3$. In other words, we may assume that $a = 0$.

We use Maple to solve for the conditions that all coefficients of r_6, r_7, and r_8 are 0, assuming that $p = 1, m = 1, a = 0$. Clearly we can also assume that we do not have $c = d = 0$, in which case r_3 would lead to a non-domain. With these constraints there is only one solution family, as follows.

Example 3.13. Let $\mathcal{H}(b)$ be the algebra with the four relations

$$r_1 : x_2 x_1 = x_1 x_2 + x_1^2$$

$$r_2 : x_3 x_2 = 2b x_1 x_3 + b x_2 x_3$$

$$r_3 : x_3^2 x_1 = -b^2 x_1 x_3^2 + 2b x_3 x_1 x_3$$

$$r_4 : x_3 x_1^2 = -b^2 x_1^2 x_3 + 2b x_1 x_3 x_1.$$

(3.13.1) $\mathcal{H}(b)$ is a graded twist of $\mathcal{H}(1)$ by the automorphism sending $x_1 \to x_1$, $x_2 \to x_2$, $x_3 \to b^{-1} x_3$.

(3.13.2) The opposite ring of $\mathcal{H}(b)$ is isomorphic to $\mathcal{H}(-b)$.

(3.13.3) $\mathcal{H}(1)$ has a normal element $y := x_3 x_1 - x_1 x_3$, and the factor ring $\mathcal{H}(1)/(y)$ is an AS-regular algebra of dimension 3.

The same argument as in Example 3.5 shows that y is a regular element, and so Lemma 1.3(b,c) applies to show that $\mathcal{H}(b)$ is AS-regular and has all good properties. The example $\mathcal{H}(b)$ can also be written as an Ore extension: If $B = k\langle x_1, x_3 \rangle / (r_3, r_4)$, then B is a cubic AS-regular algebra, and $\mathcal{H}(b) \cong B[x_2; \sigma, \delta]$ for the appropriate σ, δ. Thus Lemma 1.3(a) also applies to this example.

3.5. **Other leading terms for r_4.** We still need to consider whether we get any AS-regular algebras in which the leading term of r_4 is smaller than $x_3 x_1^2$. The answer is no, but we can only prove this through a computer calculation to exhaust the possibilities. As the method is entirely similar as in the preceding cases, we do not give all of the details of the messy coefficients that arise. The interested reader can find more details at the first author's webpage; see Remark 3.14 below.

Suppose first that $\ell = 0$ but that the leading term of r_4 is the next highest possibility $x_2 x_3 x_1$, so $k \neq 0$. We can assume that $k = -1$ and so r_4 is

$$r_4 : x_2 x_3 x_1 = (f x_1^2 + g x_1 x_2 + h x_2^2) x_3 + j x_1 x_3 x_1,$$

while the relations r_1, r_2, r_3, r_5 are unchanged. We note that an easy calculation using the multi-graded Hilbert series (E2.1.1) shows that since A is AS-regular we should have $\dim_k A_{(2,2)} = 6$ and $\dim_k A_{(3,1)} = 7$.

Now the leading terms of the relations $r_1 - r_5$ already imply that $A_{(2,2)}$ is spanned by the 6 monomials

(E3.13.1) $$\{ x_3 x_1 x_3 x_1, x_3 x_1^2 x_3, x_1 x_3 x_1 x_3, x_2^2 x_3^2, , x_1 x_2 x_3^2, x_1^2 x_3^2 \},$$

so these are a basis for $A_{(2,2)}$. The overlaps $x_3^2 x_1 x_2$ between r_5 and r_3 and $x_3 x_2 x_3 x_1$ between r_2 and r_4 produce respective relations r_6 and r_7 involving monomials in (E3.13.1), and so every coefficient of r_6 and r_7 is 0. We suppress the formulas for these relations, but mention that the coefficient in r_6 of $x_3 x_1 x_3 x_1$ is $-qba - qn + ea - ze$, just as it was in the $\ell = -1$ case (though r_6 as a whole is now different), so in particular this forces $-qba - qn + ea - ze = 0$.

We also now have two overlaps in multi-degree $(3,1)$ to resolve, $x_3 x_1 x_2 x_1$ and $x_2 x_3 x_1 x_2$. It is easy to check that the first leads to a genuine new relation r_8 with leading term $p x_3 x_1^2 x_2$. Then the given leading terms of r_1, r_2, r_4, r_5 plus the new leading term $x_3 x_1^2 x_2$ imply that $A_{(3,1)}$ is spanned by the seven monomials

$$\{x_3 x_1^3, x_1 x_3 x_1^2, x_1^2 x_3 x_1, x_2^3 x_3, x_1 x_2^2 x_3, x_1^2 x_2 x_3, x_1^3 x_3\},$$

which must be then be a basis for $A_{(3,1)}$. Resolving $x_2 x_3 x_1 x_2$ gives a new degree-$(3,1)$ relation r_9, every coefficient of which must now be 0.

We now have the same three cases for r_1 as before, and the same reductions to r_2 in each case are allowed. If $p = 1, m = 0$, we can reduce r_2 so that $a = b = 0, n = 1$. Since the coefficient $(ea - q - qba - aqe)$ of r_6 cannot be 0, we get no solutions. If $m = 0, p \neq 1$, then we can assume that $a, b \neq 0$, $n = 1$. Again in order for A to be a domain, Lemma 3.2 shows we can assume that $d \neq bc$. Solving for the coefficients of r_6, r_7, r_9 to be zero under these constraints using Maple gives no solutions. Finally, if $p = 1, m = 1$, we can adjust r_2 so that $a = 0$, and we may also assume that we do not have $c = d = 0$ (or else A will not be a domain). Setting the coefficients of r_6, r_7, r_9 to 0, there are again no solutions.

The remaining possibilities for the leading terms of r_4 are comparatively easily eliminated. If we have $\ell = 0, k = 0, h \neq 0$, we can assume that r_4 has the form

$$r_4 : x_2^2 x_3 = (f x_1^2 + g x_1 x_2)x_3 + j x_1 x_3 x_1$$

and the relations r_1, r_2, r_3, r_5 are unchanged from the previous cases. With the leading terms we have, the six monomials

$$\{x_3 x_1 x_3 x_1, x_3 x_1^2 x_3, x_1 x_3 x_1 x_3, x_2 x_3 x_1 x_3, x_1 x_2 x_3^2, x_1^2 x_3^2\}$$

span $A_{(2,2)}$ and so must be a basis. One may check that resolving the overlap $x_3 x_2^2 x_3$ between r_4 and r_2 leads to a new degree $(2,2)$-relation among these monomials $r_6 : -j x_3 x_1 x_3 x_1 + \ldots$, which forces $j = 0$. But then the relation r_4 shows that A is not a domain, since r_4 has the form $f x_3$ where f is not a relation.

The final case is where $\ell = k = h = 0$. In this case r_4 has the form $x_1 f$ for a nonzero f and A cannot be a domain.

Remark 3.14. We give some brief comments about the computational methods which were used to obtain the results above. Our main program is a simple Maple program which reduces an element of the free algebra (with coefficients involving unknown parameters) using given relations. This was used to calculate the relations r_6, r_7, etc. arising from overlaps in the diamond lemma. The Maple solve command was used

to solve the simultaneous system of equations given by setting all of the coefficients of these relations to 0. The reader might be concerned at the lack of proof that the Maple solve command really finds all solutions to such a complicated system. We did verify the computation in the most important case by checking that when $\ell \neq 0, p \neq 1, n = 1$, the system given by the coefficients of $r_6 - r_8$ can actually be solved by hand to give the same set of solutions.

We also wrote programs in Maple to calculate the matrix d_3 in the free resolution of $_Ak$ as in Lemma 3.4, and to find some normal elements of low degree, as was needed for examples $\mathcal{D} - \mathcal{G}$. The calculation of d_3 could presumably have also been accomplished by trial and error, but it would have been difficult to find the required sequences of normal elements by hand, especially in example $\mathcal{F}$ where one normal element has degree 6. Our program which reduces elements using the relations can also be used to check the output of these programs, for instance that the claimed normal elements really are.

The reader wishing to verify our computation can find the Maple code for these programs, which we make freely available, on the first author's website: http://www.math.ucsd.edu/∼drogalsk.

4. **Proof of the main results**

Our main theorems all quickly follow from the results of the last section.

Proof of Theorem 0.1. The theorem is just a summarization of the classification we did in the last section, which showed that any properly $\mathbb{Z}^{\times 2}$-graded AS-regular algebra which is a domain with three generators is either a normal extension, or else is isomorphic to one of the examples $\mathcal{A}(b, q), \mathcal{B}(b), \cdots, \mathcal{H}(b)$ (we may exclude the family $\underline{\mathcal{F}}$, so there are eight families here). Note that although we sometimes passed to the opposite ring in our reductions, our calculations of the opposite rings of our examples show that each opposite ring is always isomorphic to an algebra already on the list (in fact, an example in the same family.) The result follows. □

Proof of Theorem 0.2. By our classification, if A is a properly $\mathbb{Z}^{\times 2}$-graded noetherian AS-regular algebra with 3 generators that is not a normal extension, then again A is isomorphic to one of the algebras $\mathcal{A} - \mathcal{H}$. In section 3 we showed that each of these algebras (or some graded twist) satisfies at least one of the conditions (a),(c), or (d) in Lemma 1.3, and so by that lemma A is also strongly noetherian, Auslander regular and Cohen-Macaulay.

If A is a normal extension (of a 3-dimensional AS-regular algebra), then it is strongly noetherian, Auslander regular and Cohen-Macaulay by Lemma 1.3(c). □

Proof of Corollary 0.3. By [11, Proposition 1.4], A is generated by either 2, or 3, or 4 elements. If A is generated by 3 elements, this is Theorem 0.2. If A is generated by 2 elements, the assertion follows from [11, Theorems A, B, C]. If A is generated by 4 elements, the assertion follows from [27, Theorems 0.1 and 0.2] and Lemma 1.3(a). □

5. Graded isomorphisms and automorphisms

In this section, we study graded isomorphisms between the AS-regular algebras classified above, for generic values of the parameters. The first task is to show in the next lemma that any such graded isomorphism has a limited form; in particular, it preserves the bigrading.

Lemma 5.1. *Suppose that A and A' are any two algebras among the examples $\mathcal{A}$-$\mathcal{G}$ given in section 3, where A has relations $r_1 : x_2x_1 = px_1x_2$ and $r_2 : x_3x_2 = ax_3x_1 + x_1x_3 + bx_2x_3$, and A' has respective parameters p', a', b' in its first two relations.*

Similarly, let B and B' be any two algebras among the family $\mathcal{H}$ in section 3, where B has relations $\widehat{r}_1 : x_2x_1 = x_1x_2 + x_1^2$ and $\widehat{r}_2 : x_3x_2 = 2\beta x_1x_3 + \beta x_2x_3$ and B' has parameter β' in its second relation. Assume that $p, b, p', b', \beta, \beta' \neq 0, 1$ and $a, a' \neq 0, -1$.

(a) *If there is a graded isomorphism $\sigma : A \to A'$, one of the following two cases must occur:*

    (i) *σ is given by $x_1 \to \lambda x_1, x_2 \to \lambda x_2, x_3 \to \mu x_3$ for some $\lambda, \mu \in k^\times$; or*

    (ii) *σ is given by $x_1 \to \lambda x_2, x_2 \to \rho x_1, x_3 \to \mu x_3$ for some $\lambda, \mu, \rho \in k^\times$.*

(b) *If there is a graded isomorphism $\sigma : B \to B'$, then σ is determined by $x_1 \to \lambda x_1, x_2 \to \lambda x_2, x_3 \to \mu x_3$ for some $\lambda, \mu \in k^\times$.*

(c) *There does not exist a graded isomorphism $\sigma : A \to B$.*

Proof. (a) Any graded algebra isomorphism $\sigma : A \to A'$ is determined by $\sigma(x_i) = \sum_{j=1}^3 a_{ij}x_j$ for an invertible 3×3-matrix $(a_{ij})_{3\times 3}$. Applying σ to the relation r_1 of A and restricting to the degree $(0,2)$-piece gives rise to the equation $a_{13}a_{23} = pa_{23}a_{13}$. Since $p \neq 1$, we have $a_{13}a_{23} = 0$, which means that either $a_{13} = 0$ or $a_{23} = 0$. We assume that $a_{13} = 0$; the analysis in the other case is symmetric. Now applying σ to r_2 and considering the degree $(0,2)$-piece gives $a_{33}a_{23} = ba_{23}a_{33}$. Since $b \neq 0$, we have $a_{33} = 0$ or $a_{23} = 0$. Suppose that $a_{33} = 0$. Then $\sigma(x_1), \sigma(x_3)$ have degree $(1,0)$ in A', and so must span $kx_1 + kx_2$ in A'. Thus the relation r_1' of A' pulls back under σ to give some degree 2 relation in A involving only x_1 and x_3. Since A has no such relation, this is a contradiction. Thus $a_{23} = 0$. We conclude that $\sigma(x_1), \sigma(x_2) \in kx_1 + kx_2$, and $a_{33} \neq 0$.

Now considering the relation $\sigma(r_1)$ again and using that $p, p' \neq 1$, it is easy to prove that either (i) $\sigma(x_1) = a_{11}x_1, \sigma(x_2) = a_{22}x_2, p = p'$, or (ii) $\sigma(x_1) = a_{12}x_2, \sigma(x_2) = a_{21}x_1, p' = p^{-1}$.

Consider case (i). Looking at $\sigma(r_2)$ and restricting to degree $(2,0)$ gives a relation

$$(a_{31}x_1 + a_{32}x_2)(-aa_{11}x_1 + a_{22}x_2) = (a_{11}x_1 + ba_{22}x_2)(a_{31}x_1 + a_{32}x_2).$$

Since r_1' is the only relation in A' of degree $(2,0)$ and $a \neq -1, b \neq 1$, it is easy to see that $a_{32} = a_{31} = 0$. Finally, restricting $\sigma(r_2)$ to degree $(1,1)$ gives a relation

$$a_{33}x_3(-aa_{11}x_1 + a_{22}x_2) = (a_{11}x_1 + ba_{22}x_2)a_{33}x_3,$$

which must be a multiple of r_2'. It is easy to check that this implies that $a_{11} = a_{22}$.

The argument in case (ii) is similar and we leave it to the reader.

(b) The proof is similar to the proof of (a) and we omit it.

(c) Suppose that $\sigma : A \to B$ is a graded isomorphism. The same argument as in the first paragraph of the proof of (a) still applies to $\sigma : A \to B$, since this argument used only the form of the relations of A. So $\sigma(x_1), \sigma(x_2) \in kx_1 + kx_2$. In particular, σ restricts to an isomorphism $k\langle x_1, x_2 \rangle / (r_1) \to k\langle x_1, x_2 \rangle / (\widehat{r}_1)$. This is impossible, since the quantum plane and the Jordan plane are well-known to be non-isomorphic; or, a simple argument similar to the above easily proves this. $\qquad \square$

We call an isomorphism of the form given in part (a)(i) or part (b) of the lemma *trivial* and an isomorphism of the form given in part (a)(ii) *quasi-trivial*. The lemma now makes it easy to determine all possible isomorphisms and automorphisms of the algebras classified earlier, when the parameters are not special. To make this precise, suppose that A is one of the algebras of types $\mathcal{A} - \mathcal{H}$ given in Section 3. In order to give a uniform treatment, we include the family $\underline{\mathcal{F}}$. We say that A is *generic* if the parameters in r_1, r_2 satisfy the hypothesis of Lemma 5.1. Explicitly, this is equivalent to $b \neq 1, q^2 b \neq 1$ for type $\mathcal{A}(b, q)$, to $b \neq 1, b \neq \gamma$ for type $\mathcal{F}(b, \gamma)$ and type $\underline{\mathcal{F}}(b, \gamma)$, and to $b \neq 1$ for $\mathcal{H}(b)$ and all of the other types. Since the point of this section is to investigate isomorphisms and automorphisms among these algebras, we consider any two generic AS-regular algebras in the families $\mathcal{A} - \mathcal{H}$ on our list to be the same at the moment if and only if they have exactly the same relations; if they have different relations we call them *distinct*.

Theorem 5.2. *Consider the generic AS-regular algebras of types $\mathcal{A} - \mathcal{H}$.*

(a) *If A is a generic AS-regular algebra, the graded automorphism group of A is isomorphic either to $k^\times \times k^\times$ or to $k^\times \times k^\times \times \mathbb{Z}/(2)$. The first case occurs if A is of type $\mathcal{A}(b, q)$ with $q \neq -1$, $\mathcal{D}(h, b)$ with $h \neq b^4$, $\mathcal{F}$, $\underline{\mathcal{F}}$, or $\mathcal{H}$. The second case occurs if A is of type $\mathcal{A}(b, -1), \mathcal{B}, \mathcal{C}, \mathcal{D}(h, b)$ with $h = b^4$, $\mathcal{E}$, or $\mathcal{G}$.*

(b) *All distinct generic AS-regular algebras are pairwise non-isomorphic except for the isomorphisms given in (3.5.3), (3.8.3) and (3.11.3).*

Proof. Let A, A' be generic AS-regular algebras. By Lemma 5.1 we see that any graded isomorphism $A \to A'$ is either trivial or quasi-trivial. Since A is $\mathbb{Z}^{\times 2}$-graded, the graded automorphism group of A includes at least the trivial automorphisms $x_1 \to \lambda x_1, x_2 \to \lambda x_2, x_3 \to \mu x_3$ for any $\lambda, \mu \in k^\times \times k^\times$. Moreover, if A has any quasi-trivial automorphism σ, then clearly σ^2 is trivial. So the graded automorphism group is isomorphic to $k^\times \times k^\times$ if A has no quasi-trivial automorphism, or to $k^\times \times k^\times \times \mathbb{Z}/(2)$ if A does have a quasi-trivial automorphism.

If A is of type $\mathcal{H}$, it has no quasi-trivial isomorphism to another generic AS-regular algebra, by Lemma 5.1. Now for a given generic AS-regular algebra A of some type $\mathcal{A} - \mathcal{G}$, we claim there is exactly one generic AS-regular algebra $\widetilde{A}$ such that there exists a quasi-trivial isomorphism $A \to \widetilde{A}$. In our listing of AS-regular algebras in Section 3, we gave an explicit such isomorphism $\sigma : A \to \widetilde{A}$ in each case (in case $\mathcal{F}$ or $\underline{\mathcal{F}}$, take

(3.11.3) or its inverse, respectively.) Conversely, if $\rho : A \to A'$ is another quasi-trivial isomorphism, then $\sigma\rho^{-1} : A' \to \widetilde{A}$ is a trivial isomorphism. Precomposing with some trivial automorphism $\tau : A' \to A'$ we can get an isomorphism $\sigma\rho^{-1}\tau : A' \to \widetilde{A}$ which sends $x_i \to x_i$ for all i. This is easily seen to force A' and $\widetilde{A}$ to have exactly the same relations, proving the claim. In our listing of the AS-regular algebras in Section 3, we gave an explicit quasi-trivial automorphism of the AS-regular algebras $\mathcal{A}(b, q)$ with $q = -1$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}(h, b)$ with $h = b^4$, $\mathcal{E}$, or $\mathcal{G}$. Thus $\widetilde{A} = A$ for these algebras. On the other hand, for the other types $\mathcal{A}(b, q)$ with $q \neq -1$, $\mathcal{D}(h, b)$ with $h \neq b^4$, $\mathcal{F}$, and $\underline{\mathcal{F}}$, we gave a quasi-trivial isomorphism from A to a distinct algebra, and so $\widetilde{A} \neq A$. This finishes the proof of part (a).

To prove part (b), note that if $\sigma : A \to A'$ is a trivial isomorphism between generic AS-regular algebras, then precomposing with a trivial automorphism we see similarly as in the previous paragraph that in fact A' and A have exactly the same relations. On the other hand, if $\sigma : A \to A'$ is a quasi-trivial isomorphism, then $A' = \widetilde{A}$ as defined above, and $\widetilde{A} \neq A$ exactly for the isomorphisms in (3.5.3), (3.8.3), and (3.11.3). Note that these isomorphisms match up pairs of examples in family $\mathcal{A}$ and pairs of examples in family $\mathcal{D}$, and match up the entire family $\mathcal{F}$ with the family $\underline{\mathcal{F}}$. $\square$

We note that this final theorem justifies the remark made in the introduction that the eight families $\mathcal{A} - \mathcal{H}$ (excluding $\underline{\mathcal{F}}$) in our classification are generically pairwise non-isomorphic.

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